2013 Junior Division Second Round Solution

1. When the digits 0, 1, 2, 5, 6, 8 and 9 are rotated 180°, they become 0, 1, 2, 5, 9, 8 and 6 respectively. What does 9105 become when the four-digit number is rotated 180°?

   (A) 6150　　(B) 6102　　(C) 5016　　(D) 2019　　(E) 2016

【Solution】

The last digit 5 of the given number becomes the first digit 5 of the number we seek. The other three digits, namely, 0, 1 and 9, become 0, 1 and 6 respectively. Hence the number we seek is 5016. The answer is (C).

Answer：(C)

2. What is the value of the expression

$$(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(c+a-b)$$

   (A) 0　　(B) $a^2+b^2+c^2$　　(C) $ab+bc+ca$

   (D) $a^2+b^2+c^2-ab-bc-ca$　　(E) $a^2+b^2+c^2+ab+bc+ca$

【Solution 1】

We have $(a-b)(a+b-c)=a^2-b^2-ac+bc$, $(b-c)(b+c-a)=b^2-c^2-ab+ac$ and $(c-a)(c+a-b)=c^2-a^2-bc+ab$. These three expressions add up to 0. The answer is (A).

【Solution 2】

If we set $a=b$, we have $(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(c+a-b)=0$. Hence the expression is divisible by $(a-b)$. Similarly, it is divisible by $(b-c)$ and by $(c-a)$, so that it is a multiple of $(a-b)(b-c)(c-a)$. However, the expression itself is of second degree. It follows that the multiple must be 0. The answer is (A).

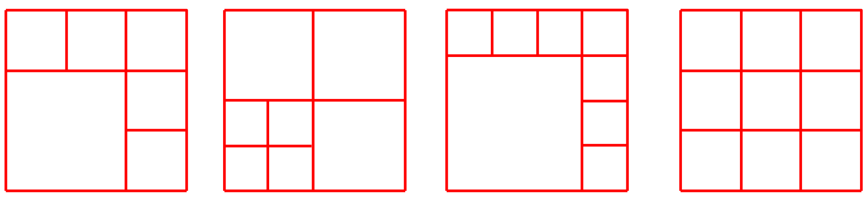
Answer：(A)

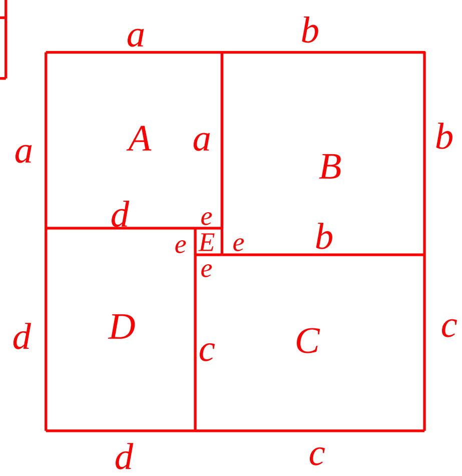
3. We wish to dissect a square into n squares which need not be of the same size. For which of the following values of n is this impossible?

   (A) 5　　(B) 6　　(C) 7　　(D) 8　　(E) 9

【Solution】

The diagram below shows that n can be 6, 7, 8 or 9.





Suppose this is also possible for $n = 5$. Clearly not all 5 can be of the same size. Moreover, each corner of the original square must belong to a different one of the smaller squares. As the figure, let the 5 small squares be A, B, C, D and E with lengths of the edge be a, b, c, d and e, respectively.

Thus $a+b=a+d$ and $b+c=a+b$. So $b=d$ and $a=c$.

And since $a+e+c=b+e+d$, $2a+e=2b+e$, i.e. $a=b$. From $a+e+c=a+b$, we can get $e=0$, which is a contradiction. Hence the square can't be dissect into 5 squares. The answer is (A).

Answer：(A)

4. The total number of players on three badminton teams is 29. No two players on the same team play against each other, while every two players on different teams play each other exactly once. What is the maximum number of games played?

(A) 265 (B) 270 (C) 276 (D) 280 (E) 282

【Solution】

Suppose team A has at least two more players than team B. Transfer one player X from team A to team B. Before the transfer, X plays every player on team B. After the transfer, X plays every player left on team A, which is still more than the number of players originally in team B. So the total number of games played has increased. It follows that to maximize the total number of games played, the size of the teams should not differ by more than 1. Hence we should have 10 players in two of the teams and 9 players in the third team. The total number of games is then $10 \times 10 + 10 \times 9 + 10 \times 9 = 280$. The answer is (D).

Answer：(D)

5. Two distinct quadratic polynomials $f(x)$ and $g(x)$ with leading coefficients equal to 1 satisfy $f(1)+f(3)+f(5)=g(1)+g(3)+g(5)$. Find all solutions of $f(x)=g(x)$.

(A) $x\le 0$ (B) $-2\le x\le 0$ (C) $0\le x\le 1$ (D) $2\le x\le 2$ (E) 3

【Solution】

Let $f(x)=x^2+ax+b$ and $g(x)=x^2+cx+d$.

Then $f(1)+f(3)+f(5)=35+9a+3b$ and $g(1)+g(3)+g(5)=35+9c+3d$.

It follows that $9(a-c)=3(d-b)$.

We now have $0=f(x)-g(x)=(a-c)x+(d-b)$.

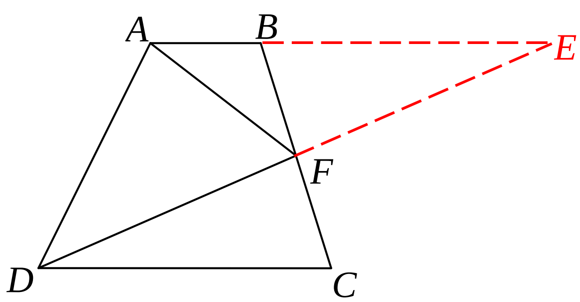
Hence $x=\dfrac{d-b}{a-c}=3$ is the unique solution.

Answer：(E)

6. The diagram shows a quadrilateral *ABCD* with *AB* parallel to *DC*. *F* is the midpoint of *BC*. If the area of triangle *AFD* is 10 cm^2, what is the area, in cm^2, of *ABCD*?

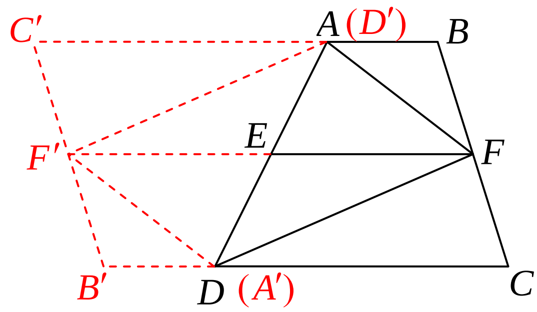
【Solution 1】

Let the extensions of *AB* and *DF* meet at *E*. Since *BF* = *CF* and *BE* is parallel to *DC*, triangles *BEF* and *CDF* are congruent so that *EF* = *DF*. Since the area of *ADF* is 10 cm^2, the area of *AEF* is also the same. It follows that the area of *ABCD* is 20 cm^2.



【Solution 2】

Let E be the midpoint of AD and connect EF. Rotate the quadrilateral $ABCD$ 180° with centre E to get a new quadrilateral $A'B'C'D'$. As the figure, coincide AD and $D'A'$ with the original quadrilateral to make up a quadrilateral $BCB'C'$.



Observe that the quadrilateral $BCB'C'$ and $AFDF'$ are parallelograms and FF' is the midsegment of $BCB'C'$. Thus the quadrilateral $BFF'C'$ and $FCB'F'$ are also parallelograms and $BC' = FF' = CB'$ and hence the area of $\triangle AFF'$ is half of the area of the parallelogram $BFF'C'$, the area of $\triangle F'FD$ is half of the area of the parallelogram $FCB'F'$. So the area of the parallelogram $AFDF'$ is half of the area of the parallelogram $BCB'C'$.

Since the area of the parallelogram $AFDF'$ is twice of the area of $\triangle AFD$, i.e. $10\times2=20$ cm^2, the area of the parallelogram $BCB'C'$ is$20\times2=40$ cm^2. So the area of the quadrilateral $ABCD$ is $40\div2=20$ cm^2.

Answer：20 cm^2

7. Lily has 2014 chocolates. She eats one on the first day. Each day after, she eats twice as many as the day before, until all the chocolates have been eaten. How many chocolates did she eat on the last day?

【Solution】

During the first ten days, Lily eats respectively 1, 2, 4, 8, 16, 32, 64, 128, 256 and 512 chocolates, Since 1+2+4+8+16+32+64+128+256+512=1024－1=1023, 2014－1023=991 chocolates are left, which is less than 1024.

Hence the number of chocolates Lily eats on the last day is 991.

Answer：991

8. Leon is putting 99 apples into boxes of two different sizes. A large box can hold 12 apples while a small box can hold 5 apples. All boxes must be full. How many boxes will he need if this number must be greater than 10?

【Solution】

Since 12 is even and 99 is odd, the number of small boxes must be odd. Since both 12 and 99 are multiples of 3, the number of small boxes must also be a multiple of 3. We have less than 20 small boxes. If we have 15 small boxes holding 75 apples, the remaining $99-75=24$ apples can go in 2 big boxes for a total of 17 boxes. If we have 9 small boxes holding 45 apples, the remaining $99-45=54$ will not completely fill the big boxes. If we have only 3 small boxes holding 15 apples, the remaining $99-15=84$ apples can go into 7 big boxes, but for a total of only 10. Hence Leon must have used 17 boxes.

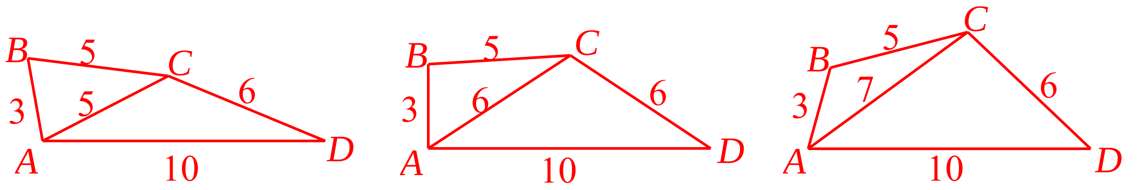
Answer：17

9. In a convex quadrilateral $ABCD$, $AB = 3$, $BC = 5$, $CD = 6$, $DA = 10$, and the length of the diagonal AC is a positive integer. How many different possible shapes can $ABCD$ take?

【Solution】

We apply the Triangle Inequality twice. In triangle ABC, we have $AC < AB + BC = 8$ and $AC > BC - AB = 2$. In triangle ADC, we have $AC < CD + DA = 16$ and $AC > DA - CD = 4$. Hence $AC = 5$, 6 or 7. Once the length of AC is determined, the shape of $ABCD$ is also determined.

Thus there are 3 different possible shapes for $ABCD$.



Answer：3 shapes

10. Divide the ten positive integers from 1 to 10 into two groups so that when the product of the numbers in the first group is divided by the product of the numbers in the second group, the quotient is a positive integer. What is the minimum value of this quotient?

【Solution】

The number 7 does not divide any of the other numbers. So it must be placed in the first group. Since it is not divisible by any of the other numbers apart from 1, the quotient is no smaller than 7. The product of the remaining nine numbers is $2^8 \times 3^4 \times 5^2$. If we can make the product of the numbers in the second group equal to $2^4 \times 3^2 \times 5^1$, then the quotient will attain its minimum value of 7. This is indeed possible, as we can put 1, 2, 4, 9 and 10 in the second group, while adding 3, 5, 6 and 8 to 7 in the first. It follows that the minimum quotient is $\dfrac{3\times5\times6\times7\times8}{1\times2\times4\times9\times10}=7$

Answer：7

11. A cardboard windmill with three blades is made from four equilateral triangles of side length 6 cm. Two triangles sharing a common vertex have the corresponding sides lying on the same straight line, as shown in the diagram. The area of the circle swept out by the blades of the windmill is x cm^2. What is the greatest integer less than or equal to x?

【Solution】

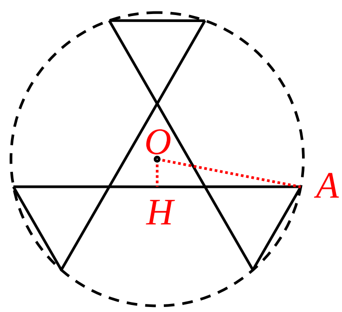
Let O be the centre of the windmill, A be a corner of one of the blades, and H be such that OH is perpendicular to AH.

Now $AH = 6+\dfrac{6}{2}=9$ cm while $OH^2=\dfrac{6^2}{12}=3$ cm^2. By Pythagoras' Theorem, $OA^2 = OH^2 + AH^2 = 3+81=84$ cm^2.

Hence the area of the circle is $\pi \times 84$ cm^2.

Since $3.141 < \pi < 3.142$, $263.844 < 84\pi < 263.928$.

Hence the desired integer is 263.



Answer：263

12. Let the real numbers a_1, a_2, a_3, a_4 and a_5 be such that $a_{n+1} = |a_n| - |a_n - 1|$ for $1 \le n \le 4$. If $a_5 = \frac{1}{2}$ and $a_1 = \frac{p}{q}$, where p and q are relatively prime positive integers, what is the value of $p + q$?

【Solution】

Let A be the point 0, B be the point 1 and P be the point a_4 on the number line.

Then $PA - PB = |a_4| - |a_4 - 1| = \frac{1}{2}$. It follows that P must lie between A and B, and we have $\frac{1}{2} = PA - PB = a_4 - (1 - a_4)$, Hence $a_4 = \frac{1 + \frac{1}{2}}{2} = \frac{3}{4}$. Redefining P to be a_3 satisfying $PA - PB = \frac{3}{4}$, we have $a_3 = \frac{1 + \frac{3}{4}}{2} = \frac{7}{8}$. Similarly, $a_2 = \frac{15}{16}$ and $a_1 = \frac{31}{32}$, so that $p + q$=63.

Answer：63

13. What is the minimum perimeter of a parallelogram which may be partitioned into 462 equilateral triangles of side length 1 cm?

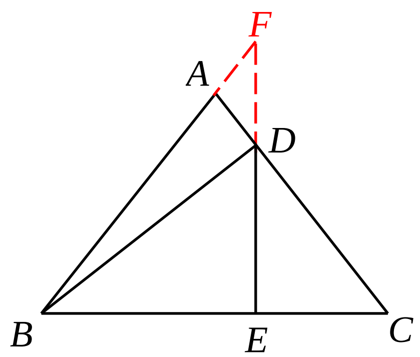
【Solution】

Clearly, the side lengths of the parallelogram are integers. Two of angles are $60°$ and the other two are $120°$. Combining the unit equilateral triangles two at a time, the parallelogram may be partitioned into 231 unit rhombuses.

Now 231=231×1=77×3= 33 × 7 = 21 × 11. Thus the minimum perimeter of the parallelogram is 2×(21+11) = 64 cm.

Answer：64 cm

14. In an acute triangle ABC, $AB = AC$. D is the foot of the perpendicular from B to CA, and E is the foot of the perpendicular from D to BC. If $BC = AB + AD$, prove that $BE = CD$.



【Solution 1】

Let the extensions of BA and ED meet at F. (5 points)

Then

$\angle AFD = 90° - \angle ABC = 90° - \angle ACB = \angle CDE = \angle ADF$

Hence $AF = AD$ so that $BC = AB + AD = BF$. (5 points)

Note that $\angle BDC = 90° = \angle BEF$ and $\angle BCD = \angle FBE$. Hence triangles BCD and FBE are congruent, so that $CD = BE$. (10 points)

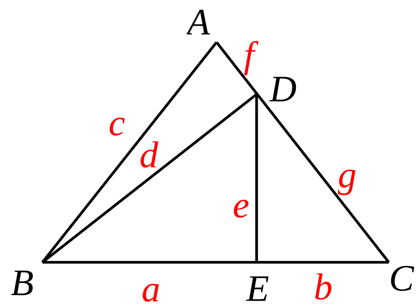
【Marking Scheme】

Extend BA to F such that BF=BC, and connect DF. Similarly, if extend BA and ED to intersect at F and show that BF=BC. (10 points).

Just extend BA and ED to intersect at a point, 0 marks.

Proved triangles BCD and FBE are congruent, so that $CD = BE$. (10 points)

【 **Solution** 2】



Let $BE=a$, $EC=b$, $AB=c$, $BD=d$, $DE=e$, $AD=f$ and $DC=g$.

Observe that $\triangle BDE\sim\triangle DCE\sim\triangle BCD$, hence we have $\frac{a}{e}=\frac{d}{g}=\frac{e}{b}$, i.e. $ab=e^2$.(5 points)

By Pythagoras' theorem, we get $c^2-f^2=d^2=a^2+e^2$. Thus

$$c^2-f^2=a^2+e^2$$
$$(c+f)(c-f)=a^2+ab$$
$$(c+f)(c-f)=a(a+b)$$

(5 points) Since $BC=AB+AD$, $a+b=c+f$.

Since $AB=AC$, $c=f+g$, i.e. $c-f=g$.(5 points)

So $(c+f)(c-f)=(a+b)g$ and hence $a=g$, i.e. $BE=DC$. (5 points)

15. A positive integer x with $n\ge 2$ digits is written down twice in a row and the $2n$-digit number so obtained is divisible by x^2. Prove that the first two digits of x are 1 and 4 in that order.

【**Solution**】

The $2n$-digit number is $(10^n+1)x$. If it is divisible by x^2, then $10^n+1=kx$ for some positive integer k. (5 points) Since $x>10^{n-1}$, $k\le\frac{10^n+1}{10^{n-1}}<11$. (5 points) Since the last digit of 10^n+1 is 1 and the sum of its digits is 2, it is not divisible by 2, 3 or 5. Hence $k=1$ or 7. (5 points) However, $k\ne 1$ as otherwise $x=10^n+1$ has $n+1$ digits. When 1000 . . . is divided by 7 until the remainder is 6, the quotient has at least three digits, the first two of which are 1 and 4 in that order. (5 points)

【Marking Scheme】

Find that $(10^n+1)x$ is divisible by x^2, and give a supposition of its quotient. Or just consider the value of the quotient. (5 points).

Find the quotient is between 1 and 10, and delete some possible values. (5 points).

Show that the quotient should be 7. (5 points).

The proof is entirely correct. (5 points).

Note:

If a student find it just need to show the quotient is 7 and prove as the quotient is 7, then the first two of the quotient are 1 and 4 in that order(10 points). Based on the clues, if he deletes some possible values to make sure the value of the quotient and only 1 or 2 values no deleted. (5 points).